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The Field Theory of Generalized Ferromagnet on the Hermitian Symmetric Spaces ¹

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Abstract

We discuss the recent developments in the generalized continuous Heisenberg ferromagnet model formulated as a nonrelativistic field theory defined on the target space of the coadjoint orbits. Hermitian symmetric spaces are special because they provide completely integrable field theories in 1+1 dimension and self-dual Chern-Simons solitons and vortices in 2+1 dimension.

Recently, an action principle of a nonrelativistic nonlinear sigma model with the target space of coadjoint orbits and its coupling with the Chern-Simons gauge field was proposed [1, 2]. The coadjoint orbits are naturally equipped with symplectic structure [3] and this can be used to construct the action for nonrelativistic field theories of generalized spins which are defined on them with arbitrary groups. The resulting models describe generalized Heisenberg ferromagnet in which the equation of motion satisfies the generalized Landau-Lifshitz (LL) equation. The Hermitian symmetric spaces [4] which are special types of the coadjoint orbits are especially interesting because they provide completely integrable field theories in 1+1 dimension [1] and self-dual Chern-Simons solitons and vortices in 2+1 dimension [2]. In this talk, I will present a review on the subject and discuss the related issues. This work was done in collaboration with Q-Han Park.

We start with a brief summary of the phase space of the coadjoint orbits and the Hermitian symmetric space. Consider a cotangent bundle $T^*G \cong G \times \mathcal{G}^*$ [5] of an arbitrary group G which can be regarded as the phase space for the generalized spin

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degrees of freedom. \mathcal{G}^* is the dual of the Lie algebra \mathcal{G} of the group G . There is a natural canonical one form

$$\theta = 2\text{Tr} (xg^{-1}dg), \quad (1)$$

and symplectic structure

$$\omega = 2\text{Tr} (xdg^{-1} \wedge dg), \quad (2)$$

where $x \in \mathcal{G}^*$ is assumed to be constant. Then, a natural symplectic left group action on T^*G can be defined by [6]

$$G \times (G \times \mathcal{G}^*) \longrightarrow G \times \mathcal{G}^* \quad (3)$$

$$(g', (g, a)) \mapsto (g'g, a). \quad (4)$$

Let us consider the associated moment map $\rho : T^*G \rightarrow \mathcal{G}^*$ via

$$\langle X, \rho(m) \rangle = m \left(\frac{d}{dt} \Big|_{t=0} \exp tX \circ g \right), \quad (5)$$

where $X \in \mathcal{G}$ and $m \in T_g^*G$ is a linear map of $\mathcal{G} \rightarrow \mathbf{R}$. Then, $\rho^A(m) \equiv \langle T^A, \rho(m) \rangle$'s, where T^A 's are the generator of \mathcal{G} , $[T^A, T^B] = f^{AB}_C T^C$ with $\text{Tr}(T^A T^B) = -1/2\eta_{AB}$, realize the Lie algebra [5]:

$$\{\rho^A, \rho^B\} = f^{AB}_C \rho^C. \quad (6)$$

It is well known that T^*G can be reduced with respect to the above momentum map by the symplectic reduction and the reduced phase space is naturally identifiable with the coadjoint orbit $\mathcal{O}_x \equiv G \cdot x \subset \mathcal{G}^*$:

$$\rho^{-1}(x)/G_x \cong G/G_x \cong G \cdot x, \quad (7)$$

where G_x is the stabilizer group of the point x . The symplectic structure (1), (2) naturally descend on the coadjoint orbit $G/G_x \equiv G/H$. Also the reduced moment map (5) becomes the generalized spin degrees of freedom which can equivalently be expressed by

$$Q = \text{Ad}^*(g)x = gxg^{-1} \quad g \in G. \quad (8)$$

Then, on the coadjoint orbit, the Eq. (5) becomes the generalizes spin algebra [5, 7];

$$\{Q^A, Q^B\} = f^{AB}_C Q^C. \quad (9)$$

An explicit example of the above procedure with an arbitrary coadjoint orbit of $G = SU(N)$ can be found in Ref. [6].

Now, we give a brief description of Hermitian symmetric spaces [4] which are special type of coadjoint orbits. A symmetric space is a coset space G/H for Lie groups whose associated Lie algebras \mathcal{G} and \mathcal{H} , with the decomposition $\mathcal{G} = \mathcal{H} \oplus \mathcal{M}$, satisfy the commutation relations,

$$[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}, \quad [\mathcal{H}, \mathcal{M}] \subset \mathcal{M}, \quad [\mathcal{M}, \mathcal{M}] \subset \mathcal{H}. \quad (10)$$

A Hermitian symmetric space is a symmetric space equipped with a complex structure. For our purpose, we need only the following properties of Hermitian symmetric spaces [4]; for each Hermitian symmetric space, there exists an element K in the Cartan subalgebra of \mathcal{G} whose centralizer in \mathcal{G} is \mathcal{H} , i.e. $\mathcal{H} = \{V \in \mathcal{G} : [V, K] = 0\}$. Also, up to a scaling, $J = \text{ad}K = [K, *]$ is a linear map $J : \mathcal{M} \rightarrow \mathcal{M}$ satisfying the complex structure condition $J^2 = \alpha$ for a constant α , or $[K, [K, M]] = \alpha M$, for $M \in \mathcal{M}$. Without loss of generality, we take α to be equal to -1 . This complex structure provides a useful identity for Q [1];

$$\begin{aligned} [Q, [Q, \partial Q]] &= g[K, [K, g^{-1}(\partial Q)g]]g^{-1} = g[K, [K, [g^{-1}\partial g, K]]]g^{-1} \\ &= -g[g^{-1}\partial g, K]g^{-1} = -\partial Q, \end{aligned} \quad (11)$$

and along with the symplectic structure, is the basic ingredient for our formulation of the integrable generalized ferromagnet. In passing, we mention that there exist six types of Hermitian symmetric spaces: $SU(p+q)/SU(p) \times SU(q)$, $SO(2n)/U(n)$, $SO(p+2)/SO(p) \times SO(2)$, $SP(n)/U(n)$ and their noncompact counterparts, and the two exceptional cases.

In application of the above, let us consider the action [1]

$$A = \int dt d^D x \text{Tr} [2Kg^{-1}\dot{g} + \partial_i(gKg^{-1})\partial_i(gKg^{-1})] \quad (i = 1, \dots, D) \quad (12)$$

where g is a map $g : R^{D+1} \rightarrow G$. The first term in the action comes from the canonical one form (1) with x replaced by the central element K . This action possesses a local H subgroup symmetry so that the physical spin variables take value on the coadjoint orbit of the Hermitian symmetric space G/H . The equations of motion can be written in terms of the generalized spin Q ,

$$\dot{Q} + \partial_i[Q, \partial_i Q] = 0, \quad (13)$$

which is the well-known homogeneous LL equation. In 1+1 dimension, the integrability of the above equation [8] arises from the zero curvature representation;

$$(\bar{\partial} - \lambda[Q, \partial Q] - \lambda^2 Q)\Psi_{HM} = 0, \quad (\partial + \lambda Q)\Psi_{HM} = 0, \quad (14)$$

where $\partial = \partial/\partial x$, $\bar{\partial} = \partial/\partial t$ and λ is an arbitrary complex constant. These linear equations are overdetermined systems whose consistency requires the integrability condition;

$$\begin{aligned} 0 &= [\bar{\partial} - \lambda[Q, \partial Q] - \lambda^2 Q, \partial + \lambda Q] \\ &= \lambda(\bar{\partial}Q + \partial[Q, \partial Q]) + \lambda^2(\partial Q + [Q, [Q, \partial Q]]). \end{aligned} \quad (15)$$

The λ^1 -order term in the second line of the above equation becomes precisely the LL equation since the λ^2 -order term vanishes identically due to the complex structure property of the Eq. (11). Having found the zero curvature representation, one can use

the well- established method [8] to calculate the infinitely many conserved quantities and soliton solutions.

One of the interesting applications of the above representation is to use the gauge equivalence of the ferromagnet and the non-linear Schrödinger (NS) model [9] to derive the generalized NS equation from the generalized LL equation. The explicit procedure and the subsequent equation can be found in Ref. [1]. In the same Reference, such a gauge equivalence was also used to relate the conserved integrals of both models and explicit conserved quantities in both models were found.

Next, we show that the reduced action of the Eq.(12) on the coadjoint orbit describes the generalized Hamiltonian dynamics [11]. In order to be explicit, we restrict to the $CP(N-1) = SU(N)/(SU(N-1) \times U(1))$ case where the element K in the Cartan subalgebra is given by $K = (i/N)\text{diag}(N-1, -1, \dots, -1)$. Now introduce a parameterization of the group element g of $SU(N)$ by an N -tuple, $g = (Z_1, Z_2, \dots, Z_N)$; $Z_p \in \mathbf{C}^N$ ($p = 1, \dots, N$), such that

$$\bar{Z}_p Z_q = \delta_{pq}, \quad \det(Z_1, Z_2, \dots, Z_N) = 1. \quad (16)$$

Then the generalized spin Q is given by

$$Q = iZ_1 \bar{Z}_1 - iI. \quad (17)$$

All other Z_p 's with $p = 2, \dots, N$ disappear in the expression of Q due to the particular form of K . In terms of the Fubini-Study coordinate ψ_α ($\alpha = 1, 2, \dots, N-1$) [10];

$$z_\alpha = \frac{\psi_\alpha}{\sqrt{1 + |\psi|^2}}, \quad z_0 = \frac{1}{\sqrt{1 + |\psi|^2}}; \quad Z_1^T = (z_0, z_1, \dots, z_{N-1}) \quad (18)$$

we have an equivalent expression of Q in component,

$$Q^A(\psi, \bar{\psi}) = -2i \sum_{p,q=0}^{N-1} \bar{z}_p (T^A)_{pq} z_q. \quad (19)$$

Substituting the above expression into the action (12), we obtain a reduced action on the $CP(N-1)$ orbit (up to a total derivative term and trivial rescaling),

$$A = \int dt dx \left(2i \frac{\bar{\psi}_\alpha \dot{\psi}_\beta}{1 + |\psi|^2} - g_{\alpha\beta} \partial \psi_\alpha \partial \bar{\psi}_\beta \right), \quad (20)$$

where $g_{\alpha\beta}$ is the Fubini-Study metric on $CP(N-1)$,

$$g_{\alpha\beta} = \frac{(1 + |\psi|^2) \delta_{\alpha\beta} - \bar{\psi}_\alpha \psi_\beta}{(1 + |\psi|^2)^2}. \quad (21)$$

Note that the first term in the above action can be written as $\int dx \theta$ where θ is the canonical one-form on $CP(N-1)$, $\theta = 2i \partial_\psi \log(1 + |\psi|^2) d\psi$ and the classical dynamics

can be described by a generalized Hamiltonian dynamics [11] with the Hamiltonian given by

$$H = \int dx g_{\alpha\beta} \partial\psi_\alpha \partial\bar{\psi}_\beta. \quad (22)$$

The Poisson bracket defined by the inverse matrix $\omega^{\alpha\beta} = -ig^{\alpha\beta}$ of the symplectic two-form $\omega = d\theta$,

$$\{F(\bar{\psi}, \psi), G(\bar{\psi}, \psi)\} = -\frac{i}{2} \int dx g^{\alpha\beta} \left(\frac{\delta F}{\delta \bar{\psi}_\alpha(x)} \frac{\delta G}{\delta \psi_\beta(x)} - \frac{\delta G}{\delta \bar{\psi}_\alpha(x)} \frac{\delta F}{\delta \psi_\beta(x)} \right) \quad (23)$$

with the inverse Fubini-Study metric $g^{\alpha\beta} = (1 + |\psi|^2)(\delta_{\alpha\beta} + \bar{\psi}_\alpha \psi_\beta)$ reproduces the generalized spin algebra, the Eq. (9) as expected. Also the Hamiltonian equation of motion gives the generalized LL equation (13);

$$\dot{Q}^A = \{H, Q^A\} = -f^{ABC} Q^B \partial^2 Q^C. \quad (24)$$

Let us consider the above model (12) in the (2+1)-dimensional case and couple with the Chern-Simons gauge fields to study the self-dual Chern-Simons solitons [12]. We introduce gauge fields A_μ which gauges the left multiplication of group G ; $g \rightarrow g'g$;

$$S = \int dt d^2x \left\{ \left[\text{Tr} (2Kg^{-1} D_t g + D_i(gKg^{-1}) D_i(gKg^{-1})) \right] - V(gKg^{-1}) + \mathcal{L}_{CS} \right\}. \quad (25)$$

The covariant derivative is defined on fundamental and adjoint representations by

$$D_\mu g = \partial_\mu g + A_\mu g, \quad A_\mu = A_\mu^A T^A, \quad D_\mu(gKg^{-1}) = D_\mu Q = [\partial_\mu + A_\mu, Q]. \quad (26)$$

The potential $V(gKg^{-1})$ is given by

$$V = \frac{1}{2} I^{AB} Q^A Q^B \quad (27)$$

where I^{AB} is a constant symmetric matrix measuring the anisotropy of the system [13]. We assume that the dynamics of gauge fields is governed by the Chern-Simons action \mathcal{L}_{CS} :

$$\mathcal{L}_{CS} = -\kappa \epsilon^{\mu\nu\rho} \text{Tr}(\partial_\mu A_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho). \quad (28)$$

Then, the equations of motion in terms of the generalized spin Q are the gauged planar LL equation;

$$D_t Q + D_i[Q, D_i Q] + [\bar{Q}, Q] = 0 \quad (29)$$

with $\bar{Q} = I^{AB} Q^A T^B$. The Gauss's law constraint is given by

$$G^A = \frac{\kappa}{2} \epsilon^{ij} F_{ij}^A - Q^A = 0. \quad (30)$$

Note that the above constraint is of a similar type with the one which appears in the well-known nonrelativistic Chern-Simons gauged NS model [14].

The Hamiltonian is given by

$$H = \int d^2x \mathcal{H} = \int d^2x \left[\frac{1}{2} (D_i Q^A)^2 + V(Q^A) \right]. \quad (31)$$

The useful identity Eq. (11), which still holds with ∂Q being replaced by $D_i Q$, brings the Hamiltonian H into the Bogomol'nyi type;

$$H = \int d^2x \left[\frac{1}{4} (D_i Q^A \pm \epsilon_{ij} [Q, D_j Q]^A)^2 + V(Q^A) \right] \pm \frac{1}{2} \epsilon_{ij} F_{ij}^A Q^A \pm 4\pi T, \quad (32)$$

where the topological charge T is defined by

$$T = \frac{1}{8\pi} \int d^2x [\epsilon_{ij} Q^A [\partial_i Q, \partial_j Q]^A - 2\epsilon_{ij} \partial_i (Q^A A_j^A)]. \quad (33)$$

Thus, the energy is bounded below by the topological charge T when the potential V is chosen such that

$$V \pm \frac{1}{2} \epsilon_{ij} F_{ij}^A Q^A = 0. \quad (34)$$

Or, upon imposing the Gauss's law constraint, it is equivalent to choosing the constant matrix

$$I^{AB} = \mp \frac{2}{\kappa} \delta_{AB}. \quad (35)$$

The minimum energy arises when the spin variable satisfies the first order self-duality equation,

$$D_i Q = \mp \epsilon_{ij} [Q, D_j Q]. \quad (36)$$

Note that with the choice (35), the potential (27) reduces to a constant. Also, in the absence of gauge fields the self-duality equation (36) is precisely that of two dimensional instantons in the principle chiral model which has been classified according to each symmetric spaces [15].

Vortices can arise in our model, if we take the gauge group to be the maximal torus subgroup of H and introduce gauge invariant terms to the action which induce vacuum symmetry breaking. Explicitly, we take $H^a (a = 1, \dots, \text{rank}(H))$ to be generators of the maximal torus group and add to the action Eq. (25) a uniform background "charge" term

$$\Delta S = \int dt d^2x A_o^a v^a, \quad (37)$$

where each v^a is a constant and the sum is taken over $a = 1, \dots, \text{rank}(H)$. Then, the gauge fields $A_\mu = A_\mu^a H^a$ and the Chern-Simons action reduces to a sum of Abelian Chern-Simons terms,

$$\mathcal{L}_{CS} = \frac{\kappa}{2} \epsilon^{\mu\nu\rho} \partial_\mu A_\nu^a A_\rho^a. \quad (38)$$

The Gauss's law is replaced by

$$\frac{\kappa}{2}\epsilon_{ij}F_{ij}^a = Q^a - v^a. \quad (39)$$

Also, we have the topological charge replacing Eq. (33)

$$T = \frac{1}{8\pi} \int d^2x [\epsilon_{ij}Q^A[\partial_i Q, \partial_j Q]^A + 2\epsilon_{ij}\partial_i((v^a - Q^a)A_j^a)]. \quad (40)$$

Assuming the potential V to be of the form

$$V(gKg^{-1}) = \frac{1}{2} \sum_a I^a (Q^a - v^a)^2, \quad (41)$$

we find that the Bogomol'nyi bound is established with the choice

$$I^1 = \dots = I^{N-1} = \mp \frac{2}{\kappa}. \quad (42)$$

Note that the potential Eq. (41) is nontrivial unlike the previous case and the non-vanishing constants v^a breaks the symmetry of the vacuum spontaneously.

Let us consider an explicit example with $CP(N-1) = SU(N)/(SU(N-1) \times U(1))$. We choose the standard expression for T^A 's: $T^A = i\lambda^A/2$ where λ^A is the $SU(N)$ Gell-Mann matrices. The Cartan subalgebra generators H^a generating the maximal torus group of $SU(N-1) \times U(1)$ are given by $N-1$ diagonal matrices

$$H_{pq}^a = i(\sum_{k=1}^a \delta_{ik}\delta_{jk} - a\delta_{i,a+1}\delta_{j,a+1})/\sqrt{2a(a+1)}; \quad a = 1, \dots, N-1. \quad (43)$$

Using the complex notation; $z = x+iy$, $\bar{z} = x-iy$, $A_z = \frac{1}{2}(A_1 - iA_2)$, $A_{\bar{z}} = \frac{1}{2}(A_1 + iA_2)$, and $D_z = \frac{1}{2}(D_1 - iD_2)$, $D_{\bar{z}} = \frac{1}{2}(D_1 + iD_2)$, we obtain an alternative expression of the self-duality equation,

$$D_z Q = \mp i[Q, D_z Q]. \quad (44)$$

With the parameterization of Q as in Eq. (19), the self-duality equation (44) for the plus sign case becomes a set of $N-1$ equations: In terms of a notation

$$D_-^\alpha \equiv \partial_z + \frac{i}{2}(A_z^1 + \frac{1}{\sqrt{3}}A_z^2 + \dots + \frac{1}{\sqrt{(\alpha-1)(\alpha-2)/2}}A_z^{\alpha-2} + \frac{\alpha}{\sqrt{\alpha(\alpha-1)/2}}A_z^{\alpha-1}), \quad (45)$$

we have

$$D_-^\alpha \bar{\psi}_\alpha = 0; \quad \alpha = 1, \dots, N-1. \quad (46)$$

Similarly, for the minus sign case, we have

$$D_+^\alpha \equiv \partial_z - \frac{i}{2}(A_z^1 + \frac{1}{\sqrt{3}}A_z^2 + \dots + \frac{1}{\sqrt{(\alpha-1)(\alpha-2)/2}}A_z^{\alpha-2} + \frac{\alpha}{\sqrt{\alpha(\alpha-1)/2}}A_z^{\alpha-1}), \quad (47)$$

and

$$D_+^\alpha \psi_\alpha = 0 ; \alpha = 1, \dots, N-1. \quad (48)$$

The Gauss's law constraint Eq. (39) is given by

$$\partial_z A_{\bar{z}}^a - \partial_{\bar{z}} A_z^a = Q^a(\psi, \bar{\psi}) - v^a. \quad (49)$$

In the $CP(1)$ case, we have only one complex ψ which we parameterize by

$$\bar{\psi} = \rho \exp(i\phi) \quad (50)$$

where ρ is real and the phase ϕ is a real multi-valued function. Then, Eq. (46) can be solved for the gauge field A and Eq. (49) reduces to a vortex-type equation;

$$\begin{aligned} A_i &= \epsilon_{ij} \partial_j \log \rho - \partial_i \phi \\ \nabla^2 \log \rho + \epsilon_{ij} \partial_i \partial_j \phi &= \frac{1}{\kappa} \left(v - \frac{1 - \rho^2}{1 + \rho^2} \right). \end{aligned} \quad (51)$$

The derivative term $\epsilon_{ij} \partial_i \partial_j \phi$ is identically zero except at the zeros of $\bar{\psi}$ where the multi-valuedness of ϕ results in the Dirac delta function(see, for example, [16]). A numerical analysis suggests that these vortex-type equations possess vortex solutions which exhibit anyonic property and show a rich structure depending on the value of v [17]. Higher N case was also treated in the Ref. [2].

In conclusion, we have shown that each Hermitian symmetric space plays an essential role in the formulation of integrable generalized Heisenberg ferromagnet in 1+1 dimension, and for the self-dual Chern-Simons solitons and vortices. It would be interesting to extend the above idea to the relativistic field theory and also to investigate the quantization problem.

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